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On the graded identities and cocharacters of the algebra of 3×3 matrices☆

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Abstract

Let $M_{2,1}(F)$ be the algebra of 3×3 matrices over an algebraically closed field F of characteristic zero with non-trivial $\mathbb{Z}_2$-grading. We study the graded identities of this algebra through the representation theory of the hyperoctahedral group $\mathbb{Z}_2 \sim S_n$. After splitting the space of multilinear polynomial identities into the sum of irreducibles under the $\mathbb{Z}_2 \sim S_n$-action, we determine all the irreducible $\mathbb{Z}_2 \sim S_n$-characters appearing in this decomposition with non-zero multiplicity. We then apply this result in order to study the graded cocharacter of the Grassmann envelope of $M_{2,1}(F)$.

Finally, using the representation theory of the general linear group, we determine all the graded polynomial identities of the algebra $M_{2,1}(F)$ up to degree 5.

Keywords: Superalgebra; Matrix; Polynomial identity; Cocharacter

1. Introduction

Let $M_k(F)$ be the algebra of $k \times k$ matrices over a field F of characteristic zero. The study of the polynomial identities satisfied by this algebra has been undertaken by several authors (see for instance [3]) but only in the case $k = 2$ quite complete results have been obtained.

In general, if an algebra A has a $\mathbb{Z}_2$-grading, then one can study its $\mathbb{Z}_2$-graded identities and then one can try to relate them to the ordinary polynomial identities

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of A. Another important reason for studying the identities of $\mathbb{Z}_2$-graded algebras is a structure theorem of Kemer asserting that any algebra A has the same identities of those of the graded tensor product of a finite dimensional $\mathbb{Z}_2$-graded algebra with the Grassmann algebra. In particular this process applied to the matrix algebras gives rise to the so-called verbally prime algebras, which are the basic building blocks in the theory of polynomial identities.

Concerning the algebra $M_k(F)$, all $\mathbb{Z}_2$-gradings have been classified in case F is algebraically closed (see [13]). In particular, it turns out that the algebra $M_3(F)$ has only one possible non-trivial $\mathbb{Z}_2$-grading and the study of the $\mathbb{Z}_2$-graded polynomial identities of this algebra is the main objective of this paper. The algebra $M_3(F)$ with this grading is usually denoted $M_{2,1}(F)$.

We study the graded identities of $M_{2,1}(F)$ through the representation theory of the hyperoctahedral group $\mathbb{Z}_2 \sim S_n$. In particular the space of multilinear polynomial identities in the first n even or odd variables has a natural structure of $\mathbb{Z}_2 \sim S_n$-module. We study the character of this module and, by splitting it into the sum of irreducibles, we determine all the irreducible $\mathbb{Z}_2 \sim S_n$-characters appearing in this decomposition with non-zero multiplicity (see [1]). We then apply this result in order to study the graded cocharacter of the Grassmann envelope of $M_{2,1}(F)$.

In the second part of the paper, using the representation theory of the general linear group, we determine all the graded polynomial identities of the algebra $M_{2,1}(F)$ up to degree 5.

2. Generalities: the $\mathbb{Z}_2 \sim S_n$-action and the $GL \times GL$-action

Throughout this paper, we shall denote by F a field of characteristic zero and by A an associative $\mathbb{Z}_2$-graded algebra (or superalgebra) over F. Recall that A is $\mathbb{Z}_2$-graded with grading $(A^{(0)}, A^{(1)})$ if A has the vector space decomposition $A = A^{(0)} \oplus A^{(1)}$ such that

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \quad \text{and} \quad A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

The elements of $A^{(0)}$ and of $A^{(1)}$ are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively.

Let $F\langle X\rangle$ be the free associative algebra on the countable set $X = \{x_1, x_2, \ldots\}$. Write $X = Y \cup Z$, the disjoint union of two sets. If we denote by $\mathscr{F}^0$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having even degree in the variables of Z and by $\mathscr{F}^1$ the subspace spanned by all monomials of odd degree in Z, then $F\langle Y \cup Z\rangle = \mathscr{F}^0 \oplus \mathscr{F}^1$ is a graded algebra with grading $(\mathscr{F}^0, \mathscr{F}^1)$.

A polynomial $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is a graded identity of A if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$.

Let $Id^{\mathrm{gr}}(A)$ denote the set of graded identities of A. Clearly $Id^{gr}(A)$ is a T_2-ideal of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms η of $F\langle X\rangle$ such that $\eta(\mathscr{F}^0) \subseteq \mathscr{F}^0$ and $\eta(\mathscr{F}^1) \subseteq \mathscr{F}^1$.

It is well known that in characteristic zero every graded identity is equivalent to a system of multilinear graded identities. Hence if we denote by

$$V_n^{gr} = \mathrm{Span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\ w_i = y_i \text{ or } w_i = z_i,\ i = 1, \dots n\}$$

the space of all multilinear polynomials of degree n in $y_1, z_1, \dots, y_n, z_n$, (i.e., y_i or z_i appears in each monomial at degree 1) the study of $Id^{gr}(A)$ is equivalent to the study of $V_n^{gr} \cap Id^{gr}(A)$, for all $n \geqslant 1$.

We act on V_n^{gr} via the hyperoctahedral group $\mathbb{Z}_2 \sim S_n$. Recall that if $\mathbb{Z}_2 = \{\pm 1\}$ is the multiplicative group of order 2 and S_n is the symmetric group on $\{1, \dots, n\}$, then $\mathbb{Z}_2 \sim S_n = \{(a_1, \dots, a_n; \sigma) \mid a_i \in \mathbb{Z}_2, \sigma \in S_n\}$ with multiplication given by

$$(a_1, \dots, a_n; \sigma)\ (b_1, \dots, b_n; \tau) = (a_1 b_{\sigma^{-1}(1)}, \dots, a_n b_{\sigma^{-1}(n)}; \sigma\tau).$$

The action of the group $\mathbb{Z}_2 \sim S_n$ on V_n^{gr} is defined as follows: for $h = (a_1, \dots, a_n; \sigma) \in \mathbb{Z}_2 \sim S_n$, $hy_i = y_{\sigma(i)}$ and $hz_i = z_{\sigma(i)}^{a_{\sigma(i)}} = z_{\sigma(i)}$ or $-z_{\sigma(i)}$ according as $a_{\sigma(i)} = 1$ or -1, respectively. For every $\mathbb{Z}_2$-graded algebra A the vector space $V_n^{gr} \cap Id^{gr}(A)$ is invariant under this action. Hence the space $V_n^{gr}(A) = V_n^{gr}/(V_n^{gr} \cap Id^{gr}(A))$ has a structure of left $\mathbb{Z}_2 \sim S_n$-module. Its character, denoted $\chi_n^{gr}(A)$, is called the nth graded cocharacter of A and the sequence $\{\chi_n^{gr}(A)\}_{n \geqslant 1}$ is the graded cocharacter sequence of A.

It is well known that in characteristic zero there is a one-to-one correspondence between irreducible $\mathbb{Z}_2 \sim S_n$-characters and pairs of partitions (λ, μ), where $\lambda \vdash r$, $\mu \vdash n - r$, for all $r = 0, \dots, n$. We also write $|\lambda| + |\mu| = n$ in this case. If $\chi_{\lambda,\mu}$ denotes the irreducible $\mathbb{Z}_2 \sim S_n$-character corresponding to (λ, μ) then, since $\mathrm{char} F = 0$, by complete reducibility we can write

$$\chi_n^{gr}(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{1}$$

where $m_{\lambda,\mu} \geqslant 0$ is the multiplicity of $\chi_{\lambda,\mu}$ in the given decomposition.

In case the algebra A is finite dimensional, say $\dim A^{(0)} = p$ and $\dim A^{(1)} = q$ then, by the superanalogue of [6, Lemma 1.2], the previous relation can be rewritten as

$$\chi_n^{gr}(A) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda) \leqslant p, h(\mu) \leqslant q}} m_{\lambda,\mu} \chi_{\lambda,\mu}, \tag{2}$$

where $h(\lambda)$ and $h(\mu)$ denote the height of the corresponding Young diagrams D_λ and D_μ, respectively.

Let $F_m\langle Y \cup Z\rangle = F\langle y_1, z_1, \dots, y_m, z_m\rangle$ denote the free associative superalgebra in m even and m odd variables and let $U = \mathrm{Span}_F\{y_1, \dots, y_m\}$, $V = \mathrm{Span}_F\{z_1, \dots, z_m\}$. The group $GL(U) \times GL(V) \cong GL_m \times GL_m$ acts naturally on the left on the space $U \oplus V$ and we can extend this action diagonally to get an action on $F_m\langle Y \cup Z\rangle$.

For every $\mathbb{Z}_2$-graded algebra A, the space $F_m\langle Y \cup Z\rangle \cap Id^{gr}(A)$ is invariant under the above action, hence the space

$$F_m(A) = F_m\langle Y \cup Z\rangle/(F_m\langle Y \cup Z\rangle \cap Id^{gr}(A))$$

has a structure of left $GL_m \times GL_m$-module. Let F_m^n be the space of all homogeneous polynomials of degree n in the variables $y_1, \dots, y_m, z_1, \dots, z_m$. Then

$$F_m^n(A) = F_m^n/(F_m^n \cap Id^{gr}(A))$$

is a $GL_m \times GL_m$-submodule of $F_m(A)$ and we denote its character by $\psi_n(A)$.

It is well known that there is a one-to-one correspondence between irreducible $GL_m \times GL_m$-characters and pairs of partitions (λ, μ), where $\lambda \vdash r$ and $\mu \vdash n - r$, for all $r = 0, \dots, n$. If we denote by $\psi_{\lambda,\mu}$ the irreducible $GL_m \times GL_m$-character associated to the pair (λ, μ), then we have the following decomposition

$$\psi_n(A) = \sum_{|\lambda|+|\mu|=n} \overline{m}_{\lambda,\mu}\psi_{\lambda,\mu}. \tag{3}$$

The $\mathbb{Z}_2 \sim S_n$-module structure of $V_n^{gr}(A)$ and the $GL_m \times GL_m$-module structure of $F_m^n(A)$ are related by the following.

Theorem 1 ([9], Theorem 3). *If the nth graded cocharacter of A has the decomposition given in* (1) *and the* $GL_m \times GL_m$*-character of* $F_m^n(A)$ *has the decomposition given in* (3) *then* $m_{\lambda,\mu} = \overline{m}_{\lambda,\mu}$, *for all* λ, μ.

It is well known (see for instance [5, Theorem 12.4.12]) that any irreducible submodule of $F_m^n(A)$ corresponding to the pair (λ, μ) is cyclic and is generated by a non-zero polynomial $f_{\lambda,\mu}$, called highest weight vector, of the form

$$\begin{aligned} &f_{\lambda,\mu}(y_1, \dots, y_p, z_1, \dots, z_q) \\ &= \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(y_1, \dots, y_{h_i(\lambda)}) \prod_{i=1}^{\mu_1} St_{h_i(\mu)}(z_1, \dots, z_{h_i(\mu)}) \sum_{\sigma\in S_n} \alpha_\sigma \sigma, \end{aligned} \tag{4}$$

where $\alpha_\sigma \in F$, the right action of S_n on $F_m^n(A)$ is defined by place permutation, $h_i(\lambda)$ (resp. $h_i(\mu)$) is the height of the ith column of the diagram D_λ (resp. D_μ) and

$$St_r(x_1, \dots, x_r) = \sum_{\tau\in S_r} (\operatorname{sgn} \tau) x_{\tau(1)} \cdots x_{\tau(r)}$$

is the standard polynomial of degree r. If $\mu = \emptyset$ then the highest weight vector corresponding to the pair $(\lambda, \emptyset)$, denoted by f_λ, is the polynomial

$$f_\lambda = f_{\lambda,\emptyset} = \prod_{i=1}^{\lambda_1} St_{h_i(\lambda)}(y_1, \dots, y_{h_i(\lambda)}) \sum_{\sigma\in S_n} \alpha_\sigma \sigma.$$

In a similar way we define f_μ, the highest weight vector corresponding to the pair $(\emptyset, \mu)$.

Let T_λ and T_μ be two Young tableaux. We denote by f_{T_λ,T_μ} the highest weight vector obtained from (4) by considering the only permutation $\tau \in S_n$ such that the integers $\tau(1), \ldots, \tau(h_1(\lambda))$, in this order, fill in from top to bottom the first column of T_λ, $\tau(h_1(\lambda)+1), \ldots, \tau(h_1(\lambda)+h_2(\lambda))$ the second column of $T_\lambda, \ldots, \tau(h_1(\lambda)+\cdots+h_{\lambda_1-1}(\lambda)+1), \ldots, \tau(r)$ the last column of T_λ; also $\tau(r+1), \ldots, \tau(r+h_1(\mu))$ fill in the first column of $T_\mu, \ldots, \tau(r+h_1(\mu)+\cdots+h_{\mu_1-1}(\mu)+1), \ldots, \tau(n)$ the last column of T_μ. As above we also define f_{T_λ} and f_{T_μ}.

The following remark is clear.

Remark 2. If

$$\psi_n(A) = \sum_{|\lambda|+|\mu|=n} \overline{m}_{\lambda,\mu}\psi_{\lambda,\mu}$$

is the $GL_m \times GL_m$-character of $F_m^n(A)$, then $\overline{m}_{\lambda,\mu} \neq 0$ if and only if there exists a pair of tableaux (T_λ, T_μ) such that the corresponding highest weight vector f_{T_λ,T_μ} is not a graded polynomial identity for A.

3. On the graded cocharacters of $M_{2,1}(F)$

In this section we study the graded cocharacters of $M_{2,1}(F)$. Recall that $M_{2,1}(F)$ is the algebra of 3×3 matrices over F with grading

$$M_{2,1}(F)^{(0)} = \begin{pmatrix} F & F & 0 \\ F & F & 0 \\ 0 & 0 & F \end{pmatrix}, \qquad M_{2,1}(F)^{(1)} = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ F & F & 0 \end{pmatrix}.$$

Recall that if A is an algebra, an ordinary polynomial identity for A is a polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$ (see [15]). In case A is $\mathbb{Z}_2$-graded, we can think of an ordinary polynomial identity as a special type of graded identity in the variables $y_1+z_1, \ldots, y_n+z_n$. Let $Id(A)$ be the ideal of ordinary polynomial identities of A. If $V_n = \mathrm{Span}_F\{x_{\sigma(1)}\cdots x_{\sigma(n)} \mid \sigma \in S_n\}$, then $V_n/(V_n \cap Id(A))$ is an S_n-module under the permutation action. The character of this module, denoted $\chi_n(A)$, is called the nth cocharacter of A.

If A and B are two algebras then it is well known and easy to see that $Id(A \oplus B) = Id(A) \cap Id(B)$. Hence the following remark is clear.

Remark 3. $M_{2,1}(F)^{(0)} \cong M_2(F) \oplus F$. Therefore $Id(M_{2,1}(F)^{(0)}) = Id(M_2(F))$.

Since $\dim_F M_{2,1}(F)^{(0)} = 5$ and $\dim_F M_{2,1}(F)^{(1)} = 4$, by relation (2) we have the following decomposition

$$\chi_n^{gr}(M_{2,1}(F)) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leqslant 5, h(\mu)\leqslant 4}} m_{\lambda,\mu}\chi_{\lambda,\mu}. \tag{5}$$

We start by considering only the multiplicities $m_{\lambda,\mu}$ with $\mu = \emptyset$. Since $Id(M_{2,1}(F)^{(0)}) = Id(M_2(F))$, if we denote by $\chi_n(M_2(F)) = \sum_{\substack{\lambda\vdash n \\ h(\lambda)\leqslant 4}} \tilde{m}_\lambda \chi_\lambda$ the ordinary nth cocharacter of $M_2(F)$, we have that $\tilde{m}_\lambda = m_{\lambda,\emptyset}$, for all $\lambda \vdash n$.

The multiplicities in $\chi_n(M_2(F))$ were computed by Formanek [8] and Drensky [4]. From an easy inspection it follows that $\tilde{m}_\lambda \neq 0$ for all $\lambda \vdash n$ except when $\lambda = (1^4)$. In fact this last case corresponds to the fact that the standard polynomial of degree 4 vanishes in $M_2(F)$. Hence we have the following.

Lemma 4. *In the nth graded cocharacter of* $M_{2,1}(F)$, *given in* (5), *we have that* $m_{\lambda,\emptyset} = 0$ *if and only if either* $h(\lambda) = 5$ *or* $\lambda = (1^4)$.

A main tool in what follows is a technique of "gluing" horizontally two Young diagrams given in [14] that we now describe.

Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash s$, $\mu = (\mu_1, \ldots, \mu_m) \vdash t$ and let D_λ, D_μ be the corresponding Young diagrams. If the last column of D_λ is larger than the first column of D_μ then we can glue D_μ to the right of D_λ, getting a new Young diagram which we denote by $D_\lambda \mid D_\mu$. If we indicate by $\lambda * \mu = (\lambda_1 + \mu_1, \ldots, \lambda_i + \mu_i, \ldots) \vdash s + t$, then $D_\lambda \mid D_\mu = D_{\lambda*\mu}$.

We can also glue Young tableaux: let $T_\lambda = D_\lambda(a_{ij})$ and $T_\mu = D_\mu(b_{hk})$ be two Young tableaux on D_λ and D_μ, respectively. Let $T_\mu + s = D_\mu(b_{hk} + s)$. We glue T_λ and $T_\mu + s$ in a similar way: if the last column of D_λ is larger than the first column of D_μ, we obtain a new tableau on $D_{\lambda*\mu}$, denoted by $T_{\lambda*\mu} = D_\lambda(a_{ij}) \mid D_\mu(b_{hk} + s)$, by gluing $T_\mu + s$ to the right of T_λ.

By the result in [14, Corollary 1.5] we can easily determine $f_{T_{\lambda*\mu}}$. In fact we have.

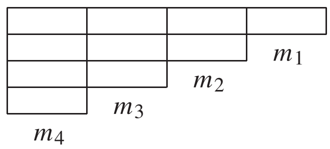
Remark 5. $f_{T_{\lambda*\mu}} = f_{T_\lambda} f_{T_\mu}$.

Now we consider only the multiplicities $m_{\lambda,\mu}$ in (5) with $\lambda = \emptyset$. We have the following result.

Lemma 6. *In the nth graded cocharacter of* $M_{2,1}(F)$ *given in* (5), $m_{\emptyset,\mu} \neq 0$ *for all* $\mu \vdash n$, $h(\mu) \leqslant 4$.

Proof. By Theorem 1 and Remark 2 it is enough to show that, for any $\mu \vdash n$, there exists a tableau T_μ such that the corresponding highest weight vector

$f_{T_\mu}(z_1, \dots, z_{h(\mu)})$ is not a graded polynomial identity for $M_{2,1}(F)$. Now, for $\mu \vdash n$ we can write D_μ in the following form

$D_\mu =$ 

where $m_i \geqslant 0$ is the number of columns of D_μ having height i. We decompose D_μ into a finite number t of diagrams D_{μ^i} each having only one column. Let T_{μ^i} be the standard tableau of D_{μ^i} obtained by filling the boxes downward with the numbers $1, 2, \dots, h(\mu^i)$, where $h(\mu^i)$ is the height of the ith column of D_μ. Then $f_{T_{\mu^i}}(z_1, \dots, z_{h(\mu^i)}) = St_{h(\mu^i)}(z_1, \dots, z_{h(\mu^i)})$. From Remark 5 we have the following decomposition

$$\begin{aligned} f_{T_\mu}(z_1, \dots, z_{h(\mu)}) &= f_{T_{\mu^1} * T_{\mu^2} * \cdots * T_{\mu^t}}(z_1, \dots, z_{h(\mu)}) \\ &= f_{T_{\mu^1}}(z_1, \dots, z_{h(\mu)}) \cdots f_{T_{\mu^t}}(z_1, \dots, z_{h(\mu^t)}) \\ &= St_{h(\mu)}^{m_{h(\mu)}}(z_1, \dots, z_{h(\mu)}) \cdots St_2^{m_2}(z_1, z_2) z_1^{m_1}. \end{aligned}$$

Now we exhibit a non-zero evaluation of $f_{T_\mu}(z_1, \dots, z_{h(\mu)})$. Consider $M_1 = e_{13} + e_{31}$, $M_2 = e_{31} + e_{32}$, $M_3 = e_{32} + e_{23}$, $M_4 = e_{23}$, where the e_{ij}'s are the usual matrix units. A direct computation shows that

$$f_{T_\mu}(M_1, \dots, M_{h(\mu)}) = \sum_{i,j=1}^{3} \alpha_{ij} e_{ij} \neq 0, \tag{6}$$

where for all possible values of $m_1, \dots, m_{h(\mu)}$, $h(\mu) \leqslant 4$, there exist indices $i, j \in \{1, 2, 3\}$ such that $\alpha_{1i} \neq 0$ and $\alpha_{3j} \neq 0$. Hence $m_{\emptyset,\mu} \neq 0$ for any $\mu \vdash n$. $\square$

We now recall a result about the values of a polynomial in a given algebra.

For an F-algebra A, we denote by $[A, A]$ the additive subgroup of A spanned by all commutators $[a, b]$, where $a, b \in A$. Also, given a polynomial $f = f(x_1, \dots, x_n) \in F\langle X \rangle$, let us denote by $f(A)$ the linear span of all valuations $f(a_1, \dots, a_n)$ with $a_1, \dots, a_n \in A$. The following result holds.

Lemma 7. *Let A be an F-algebra and let $f = f(x_1, \dots, x_n)$ be a polynomial in $F\langle X \rangle$ which is not an identity of A. Then $f(A)$ is a Lie ideal of A. If A is a simple algebra then either $f(A) \subseteq Z(A)$, the center of A, or $f(A) \supseteq [A, A]$.*

The proof of the previous lemma can be found for instance in [11] and [10].

Now we consider the multiplicities in (5) with both $\lambda \neq \emptyset$ and $\mu \neq \emptyset$.

Lemma 8. *For all $\lambda \neq \emptyset$, $\mu \neq \emptyset$, with $h(\lambda)$, $h(\mu) \leqslant 4$, we have that $m_{\lambda,\mu} \neq 0$.*

Proof. Let $\lambda \vdash r$. Suppose first that $\lambda \neq (1^4)$. Since, by Lemma 4, $m_{\lambda,\emptyset} \neq 0$, there exists a tableau T_λ such that the corresponding highest weight vector $f_{T_\lambda}(y_1, \ldots, y_{h(\lambda)})$ is not an identity for $M_{2,1}(F)^{(0)}$. Since $M_{2,1}(F)^{(0)}$ is an associative subalgebra of $M_{2,1}(F)$, then, by the previous lemma, $f_{T_\lambda}(M_{2,1}(F)^{(0)})$ is a Lie ideal of $M_{2,1}(F)^{(0)}$. Moreover, being $M_{2,1}(F)^{(0)}$ a simple algebra, it follows that either $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$. Notice that in case

$$f_{T_\lambda}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}],$$

then $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \ni e_{11} - e_{22} = [e_{12}, e_{21}]$. Hence, since f_{T_λ} is not an identity of $M_{2,1}(F)^{(0)}$ in both cases there exist elements $a_{i_1}, \ldots, a_{i_{h(\lambda)}} \in M_2(F)$, (see Remark 3), $i = 1, \ldots, s$, such that

$$\sum_{i=1}^{s} \alpha_i f_{T_\lambda}(a_{i_1}, \ldots, a_{i_{h(\lambda)}}) = e_{11} \pm e_{22}, \tag{7}$$

where we take the $+$ or the $-$ sign according as $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$, respectively. If T_μ is the Young tableau constructed in the proof of Lemma 6 with the integers $r+1, \ldots, n$, and $M_1, \ldots, M_{h(\mu)}$ are the 3×3 matrices introduced in the same proof, it follows that

$$\sum_{i=1}^{s} \alpha_i f_{T_\lambda, T_\mu}(a_{i_1}, \ldots, a_{i_{h(\lambda)}}, M_1, \ldots, M_{h(\mu)})$$
$$= \sum_{i=1}^{s} \alpha_i f_{T_\lambda}(a_{i_1}, \ldots, a_{i_{h(\lambda)}}) f_{T_\mu}(M_1, \ldots, M_{h(\mu)}).$$

From (6) and (7) we obtain that

$$\sum_{i=1}^{s} \alpha_i f_{T_\lambda}(a_{i_1}, \ldots, a_{i_{h(\lambda)}}) f_{T_\mu}(M_1, \ldots, M_{h(\mu)}) = (e_{11} \pm e_{22}) \left(\sum_{i,j=1}^{3} \alpha_{ij} e_{ij} \right) \neq 0.$$

For this it follows that there exist $a_{i_1}, \ldots, a_{i_{h(\lambda)}} \in M_2(F)$ such that

$$f_{T_\lambda, T_\mu}(a_{i_1}, \ldots, a_{i_{h(\lambda)}}, M_1, \ldots, M_{h(\mu)}) \neq 0$$

and we are done in this case.

Suppose finally that $\lambda = (1^4)$. Then $\mu \vdash n - 4$. In this case we write $D_\mu = D_{\overline{\mu}} \mid D_{\overline{\overline{\mu}}}$, where $D_{\overline{\mu}}$ consists of the first column of D_μ and $D_{\overline{\overline{\mu}}}$ consists of the remaining ones. Since $1 \leqslant h(\mu) \leqslant 4$, $\overline{\mu}$ will be equal to one of the following partitions

$$\overline{\mu}^1 = (1),\ \overline{\mu}^2 = (1^2),\ \overline{\mu}^3 = (1^3),\ \overline{\mu}^4 = (1^4).$$

Consider the following pairs of tableaux:

$$\left(T_\lambda^1 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 5\\\hline\end{array},\quad T_{\overline{\mu}^1} = \begin{array}{|c|}\hline 4\\\hline\end{array} \right),\qquad \left(T_\lambda^2 = \begin{array}{|c|}\hline 1\\\hline 4\\\hline 5\\\hline 6\\\hline\end{array},\quad T_{\overline{\mu}^2} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} \right),$$

$$\left(T_\lambda^3 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 7\\\hline\end{array},\quad T_{\overline{\mu}^3} = \begin{array}{|c|}\hline 4\\\hline 5\\\hline 6\\\hline\end{array} \right),\qquad \left(T_\lambda^4 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 6\\\hline\end{array},\quad T_{\overline{\mu}^4} = \begin{array}{|c|}\hline 4\\\hline 5\\\hline 7\\\hline 8\\\hline\end{array} \right)$$

and, for $i = 1, \ldots, 4$, let $T_{\overline{\overline{\mu}}^i}$ be filled with the remaining integers as in Lemma 6. If $h(\mu) = i$, then let $T_\mu = T_{\overline{\mu}^i * \overline{\overline{\mu}}^i}$ and $T_\lambda = T_\lambda^i$. From Remark 5 we obtain

$$\begin{aligned} & f_{T_\lambda, T_\mu}(y_1, \ldots, y_4, z_1, \ldots, z_{h(\mu)}) \\ & \quad = f_{T_\lambda^i, T_{\overline{\mu}^i}}(y_1, \ldots, y_4, z_1, \ldots, z_{h(\mu)}) f_{T_{\overline{\overline{\mu}}^i}}(z_1, \ldots, z_{h(\overline{\overline{\mu}})}) \\ & \quad = f_{T_\lambda^i, T_{\overline{\mu}^i}}(y_1, \ldots, y_4, z_1, \ldots, z_{h(\mu)}) St_{h(\overline{\overline{\mu}})}^{m_{h(\overline{\overline{\mu}})}}(z_1, \ldots, z_{h(\overline{\overline{\mu}})}) \cdots St_2^{m_2}(z_1, z_2) z_1^{m_1}, \end{aligned}$$

where m_j is the number of column of $T_{\overline{\overline{\mu}}}$ having height j. Consider $N_1 = e_{11}$, $N_2 = e_{21}$, $N_3 = e_{12}$, $N_4 = (e_{22} + e_{33})$. It is easy to check by direct computation that

$$f_{T_\lambda, T_\mu}(N_1, \ldots, N_4, M_1, \ldots, M_{h(\mu)}) \neq 0,$$

where $M_1, \ldots, M_{h(\mu)}$ are the above matrices. Thus $m_{\lambda,\mu} \neq 0$ also in this case and we are done. □

Lemma 9. *If $\mu \neq \emptyset$, $h(\mu) \leqslant 4$, we have that $m_{(1^5),\mu} \neq 0$ if and only if $\mu \neq (1)$.*

Proof. Let $\lambda = (1^5)$. If $\mu = (1)$ then, for any pair of tableaux (T_λ, T_μ), we claim that f_{T_λ, T_μ} is a graded identity of $M_{2,1}(F)$. In fact if $T_\mu = \begin{array}{|c|}\hline i\\\hline\end{array}$, and $i = 1, 2, 5$ or 6, then f_{T_λ, T_μ} is a consequence of $St_4(y_1, \ldots, y_4)$ and, so, an identity of $M_{2,1}(F)$. In case $i = 3$ or 4, since f_{T_λ, T_μ} is multilinear, it is enough to evaluate it on the e_{ij}'s. But, in this case, since z evaluates to e_{i3} or e_{3i} in order to get a non-zero value, two y_i's should evaluate to e_{33} and then the polynomial becomes zero being alternating in the y_i's. This means (see Remark 2) that $m_{(1^5),(1)} = 0$.

Suppose now that $\mu = (t)$, $t \geqslant 2$. In this case we write $D_\mu = D_{\overline{\mu}} \mid D_{\overline{\overline{\mu}}}$, where $D_{\overline{\mu}}$ consists of the first two columns of D_μ and $D_{\overline{\overline{\mu}}}$ consists of the remaining ones. Consider the following pair of tableaux:

$$\left(T_\lambda = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 5\\\hline 7\\\hline\end{array},\quad T_{\overline{\mu}} = \begin{array}{|c|c|}\hline 4 & 6\\\hline\end{array} \right)$$

and let $T_{\overline{\overline{\mu}}}$ be filled with the remaining integers in increasing order from left to right. Then $T_\mu = T_{\overline{\mu} * \overline{\overline{\mu}}}$ and by Remark 5 we obtain

$$\begin{aligned} f_{T_\lambda,T_\mu}(y_1,\dots,y_5,z_1) &= f_{T_\lambda,T_{\overline{\mu}}}(y_1,\dots,y_5,z_1) f_{T_{\overline{\overline{\mu}}}}(z_1) \\ &= f_{T_\lambda,T_{\overline{\mu}}}(y_1,\dots,y_5,z_1) z_1^{m_1}, \end{aligned}$$

where m_1 is the number of column of $T_{\overline{\overline{\mu}}}$. Consider $N_1 = e_{11}$, $N_2 = e_{21}$, $N_3 = e_{12}$, $N_4 = e_{22}$, $N_5 = e_{33}$. By direct computation one checks that

$$f_{T_\lambda,T_\mu}(N_1,\dots,N_5,M_1) \neq 0,$$

where $M_1 = e_{13} + e_{31}$. Thus $m_{(1^5),(t)} \neq 0$.

In order to complete the proof we only need to check the case $\mu \neq (t)$, $t \geqslant 1$.In this case we write $D_\mu = D_{\overline{\mu}} \mid D_{\overline{\overline{\mu}}}$, where $D_{\overline{\mu}}$ consists of the first column of D_μ and $D_{\overline{\overline{\mu}}}$ consists of the remaining ones. Consider the following pairs of tableaux:

$$\left(T_\lambda^1 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 5\\\hline 7\\\hline\end{array}, \quad T_{\overline{\mu}^1} = \begin{array}{|c|}\hline 4\\\hline 6\\\hline\end{array} \right), \quad \left(T_\lambda^2 = \begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline 6\\\hline 7\\\hline\end{array}, \quad T_{\overline{\mu}^2} = \begin{array}{|c|}\hline 2\\\hline 4\\\hline 8\\\hline\end{array} \right),$$

$$\left(T_\lambda^3 = \begin{array}{|c|}\hline 1\\\hline 4\\\hline 6\\\hline 8\\\hline 9\\\hline\end{array}, \quad T_{\overline{\mu}^3} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline 7\\\hline\end{array} \right)$$

and, for $i = 1,\dots,3$, let $T_{\overline{\overline{\mu}}^i}$ be filled with the remaining integers as in the previous lemma. By direct computation it is easy to check that

$$\begin{aligned} & f_{T_\lambda,T_\mu}(N_1,\dots,N_5,M_1,\dots,M_{h(\mu)}) \\ &= f_{T_\lambda^i,T_{\overline{\mu}^i}}(N_1,\dots,N_5,M_1,\dots,M_{h(\mu)}) St_{h(\overline{\overline{\mu}})}^{m_{h(\overline{\overline{\mu}})}}(M_1,\dots,M_{h(\overline{\overline{\mu}})}) \cdots M_1^{m_1} \neq 0, \end{aligned}$$

where i is the height of D_μ, m_j is the number of column of $T_{\overline{\overline{\mu}}^i}$ having height i and $M_1,\dots,M_{h(\mu)}$, $N_1,\dots,N_5$ are the above matrices. Thus $m_{(1^5),\mu} \neq 0$ also in this case and we are done. $\square$

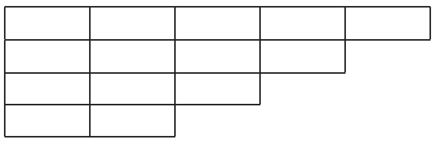
Now we prove the main result of this section.

Theorem 10. *If*

$$\chi_n^{gr}(M_{2,1}(F)) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leqslant 5, h(\mu)\leqslant 4}} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

is the nth graded cocharacter of $M_{2,1}(F)$, *then* $m_{\lambda,\mu} = 0$ *if and only if* (λ, μ) *is equal to* $((1^4), \emptyset)$ *or to* $((1^5), (1))$ *or to* $(\lambda, \emptyset)$ *with* $h(\lambda) = 5$.

Proof. If $\mu = \emptyset$ we have only even variables and, from Lemma 4, we obtain that $m_{\lambda,\emptyset} \neq 0$ if and only if $h(\lambda) \leqslant 4$ and $\lambda \neq (1^4)$. If $\lambda = \emptyset$ we have the odd case and, by applying Lemma 6, we get $m_{\emptyset,\mu} \neq 0$ for any $\mu \vdash n$, $h(\mu) \leqslant 4$. Hence we may assume that both $\lambda \neq \emptyset$ and $\mu \neq \emptyset$. If $h(\lambda) \leqslant 4$, by Lemma 8, we obtain that $m_{\lambda,\mu} \neq 0$ for all λ and μ with $h(\mu) \leqslant 4$. Hence we may assume that $h(\lambda) = 5$ and, by Lemma 9, that $\lambda \neq (1^5)$. Let $\lambda \vdash r$, $\mu \vdash n - r$. We divide the diagram D_λ horizontally into two blocks, D_{λ^1} and D_{λ^2}, contained into two strips of height 4 and 1, respectively:

$D_{\lambda^1} =$ 

$D_{\lambda^2} =$ 

We shall split the proof into four distinct cases.

Case 1. The number of boxes of D_μ is odd.

Let T_μ be the Young tableau built as in the proof of Lemma 6 with the integers $\lambda_5 + 1, \ldots, \lambda_5 + n - r$. Then, if $M_1, \ldots, M_{h(\mu)}$ are the matrices introduced above, since D_μ has an odd number of boxes,

$$f_{T_\mu}(M_1, \ldots, M_{h(\mu)}) = \sum_{i,j=1}^{3} \alpha_{ij} e_{ij} \in M_{2,1}(F)^{(1)} \tag{8}$$

and by (6) we have that $\alpha_{13} \neq 0$ and $\alpha_{3j} \neq 0$ for some $j \in \{1, 2\}$. Let T_{λ^2} be the Young tableau obtained by filling the boxes of D_{λ^2} with the integers $1, \ldots, \lambda_5$ in increasing order from left to right. Since $\lambda^1 \neq (1^4)$ and $h(\lambda^1) = 4$, by Lemma 4, $m_{\lambda^1,\emptyset} \neq 0$. Then there exists a tableau T_{λ^1} containing the integers $\lambda_5 + n - r + 1, \ldots, n$ such that the corresponding highest weight vector $f_{T_{\lambda^1}}(y_1, \ldots, y_4)$ is not a graded identity for $M_{2,1}(F)$. Then, as in the proof of Lemma 8, either $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$. In both cases, since $f_{T_{\lambda^1}}$ is not a graded identity of $M_{2,1}(F)$, there exist elements $a_{j_1}, \ldots, a_{j_4} \in M_2(F)$, $j = 1, \ldots, s$, such that

$$\sum_{j=1}^{s} \beta_j f_{T_{\lambda^1}}(a_{j_1}, \ldots, a_{j_4}) = e_{11} \pm e_{22}, \tag{9}$$

where we consider the $+$ or the $-$ sign according as $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)})$ is central or contains the commutator ideal, respectively. Then, from (8) and (9), we get

$$\begin{aligned}&\textstyle\sum_{j=1}^{s}\beta_j f_{T_\lambda,T_\mu}(e_{33},a_{j_1},\dots,a_{j_4},M_1,\dots,M_{h(\mu)})\\&\quad=e_{33}f_{T_\mu}(M_1,\dots,M_{h(\mu)})\textstyle\sum_{j=1}^{s}\beta_j f_{T_{\lambda^1}}(a_{j_1},\dots,a_{j_4})\\&\quad=e_{33}\textstyle\sum_{i,j=1}^{3}\alpha_{ij}e_{ij}(e_{11}\pm e_{22})\neq 0.\end{aligned}$$

For this it follows that there exist $a_{j_1},\dots,a_{j_4}\in M_2(F)$ such that

$$f_{T_\lambda,T_\mu}(e_{33},a_{j_1},\dots,a_{j_4},M_1,\dots,M_{h(\mu)})\neq 0$$

and we are done in this case.

Case 2. The number of boxes of D_μ is even and D_μ has at least one column of height 1.

We write $D_\mu=D_{\overline{\mu}}|D_{\overline{\overline{\mu}}}$, where $D_{\overline{\overline{\mu}}}$ consists only of the last column of D_μ. Let $T_{\overline{\overline{\mu}}}$ be the Young tableau containing the integer 1 and $T_{\overline{\mu}}$ the Young tableau built as above with the integers $\lambda_5+2,\dots,\lambda_5+n-r$. Moreover let T_{λ^2} be the Young tableau obtained by filling the boxes of D_{λ^2} with the integers $2,\dots,\lambda_5+1$ in increasing order from left to right and let T_{λ^1} be the above Young tableau. Since $D_{\overline{\mu}}$ has an odd number of boxes, by (8) and (9) we get

$$\begin{aligned}&\textstyle\sum_{j=1}^{s}\beta_j f_{T_\lambda,T_\mu}(e_{33},a_{j_1},\dots,a_{j_4},M_1,\dots,M_{h(\mu)})\\&\quad=M_1e_{33}f_{T_{\overline{\mu}}}(M_1,\dots,M_{h(\mu)})\textstyle\sum_{j=1}^{s}\beta_j f_{T_{\lambda^1}}(a_{j_1},\dots,a_{j_4})\\&\quad=M_1e_{33}\textstyle\sum_{i,j=1}^{3}\alpha_{ij}e_{ij}(e_{11}\pm e_{22})\neq 0.\end{aligned}$$

This says that there exist $a_{j_1},\dots,a_{j_4}\in M_2(F)$ such that

$$f_{T_\lambda,T_\mu}(e_{33},a_{j_1},\dots,a_{j_4},M_1,\dots,M_{h(\mu)})\neq 0$$

and we are done in this case too.

Case 3. D_μ has an even number of boxes and at least one column of height 2 or 3.

Let T_μ be the Young tableau constructed as above with the integers $\lambda_5+1,\dots,\lambda_5+n-r-1,n$, T_{λ^2} the Young tableau containing the integers $1,\dots,\lambda_5$ in this order, from left to right. Let also T_{λ^1} be a Young tableau containing the integers $\lambda_5+n-r,\dots,n-1$ such that the corresponding highest weight vector $f_{T_{\lambda^1}}(y_1,\dots,y_4)$ is not a graded identity for $M_{2,1}(F)$. Then, as we have seen, this implies that either $f_{T_\lambda^1}(M_{2,1}(F)^{(0)})\subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)})\supseteq[M_{2,1}(F)^{(0)},M_{2,1}(F)^{(0)}]$.

Suppose first that $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)})\subseteq Z(M_{2,1}(F)^{(0)})$. Let $a_{j_1},\dots,a_{j_4}\in M_2(F)$, $j=1,\dots,s$, be such that

$$\sum_{j=1}^{s}\beta_j f_{T_{\lambda^1}}(a_{j_1},\dots,a_{j_4})=e_{11}+e_{22}. \tag{10}$$

Then, by direct inspection one checks that

$$\sum_{j=1}^{s}\beta_j f_{T_\lambda,T_\mu}(e_{33},a_{j_1},\dots,a_{j_4},M_1,\dots,M_{h(\mu)})\neq 0.$$

This says that there exist $a_{j_1}, \dots, a_{j_4} \in M_2(F)$ such that

$$f_{T_\lambda, T_\mu}(e_{33}, a_{j_1}, \dots, a_{j_4}, M_1, \dots, M_{h(\mu)}) \neq 0$$

and we are done in case $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$.

Suppose now that $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$.

In particular $f_{T_{\lambda^1}}(M_{2,1}(F)^{(0)}) \ni e_{11} - e_{22} + e_{12} = [e_{12}, e_{21}] + [e_{12}, e_{22}]$. Then there exist $a_{i_1}, \dots, a_{i_4} \in M_2(F)$, $i = 1, \dots, s$, such that

$$\sum_{i=1}^{s} \alpha_i f_{T_{\lambda^1}}(a_{i_1}, \dots, a_{i_4}) = e_{11} - e_{22} + e_{12}$$

and a direct inspection shows that

$$\textstyle\sum_{i=1}^{s} \alpha_i f_{T_\lambda, T_\mu}(e_{33}, a_{i_1}, \dots, a_{i_4}, M_1, \dots, M_{h(\mu)}) \neq 0.$$

Thus there exist $a_{i_1}, \dots, a_{i_4} \in M_2(F)$ such that

$$f_{T_\lambda, T_\mu}(e_{33}, a_{i_1}, \dots, a_{i_4}, M_1, \dots, M_{h(\mu)}) \neq 0$$

and we are done in this case too.

Case 4. D_μ has an even number of boxes and only columns of height 4.

Let T_μ be the Young tableau constructed as above with the integers $\lambda_5 + 1, \dots, \lambda_5 + 1 + 4(\mu_1 - 1), \lambda_5 + 1 + 4(\mu_1 - 1) + (r - \lambda_5) + 1, \dots, n$ in this order, T_{λ^2} the Young tableau containing the integers $1, \dots, \lambda_5$ from left to right. Let also T_{λ^1} be the Young tableau containing the integers $\lambda_5 + 1 + 4(\mu_1 - 1) + 1, \dots, \lambda_1 + 1 + 4(\mu_1 - 1) + (r - \lambda_5)$ such that $f_{T_{\lambda^1}}$ is not a graded identity of $M_{2,1}(F)$. Then, this implies that either $f_{T_\lambda^1}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_\lambda^1}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$. As we have seen, in both cases there exist $a_{i_1}, \dots, a_{i_4} \in M_2(F)$, $i = 1, \dots, s$, such that

$$\sum_{i=1}^{s} \alpha_i f_{T_\lambda^1}(a_{i_1}, \dots, a_{i_4}) = e_{11} \pm e_{22},$$

where we take the $+$ or the $-$ sign according as $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \subseteq Z(M_{2,1}(F)^{(0)})$ or $f_{T_\lambda}(M_{2,1}(F)^{(0)}) \supseteq [M_{2,1}(F)^{(0)}, M_{2,1}(F)^{(0)}]$, respectively.

A direct computation shows that

$$\sum_{i=1}^{s} \alpha_i f_{T_\lambda, T_\mu}(e_{33}, a_{i_1}, \dots, a_{i_4}, M_1, \dots, M_{h(\mu)}) \neq 0.$$

For this there exist $a_{i_1}, \dots, a_{i_4} \in M_2(F)$ such that

$$f_{T_\lambda, T_\mu}(e_{33}, a_{i_1}, \dots, a_{i_4}, M_1, \dots, M_{h(\mu)}) \neq 0$$

and the proof is complete. □

4. On the graded cocharacters of $M_{2,1}(G)$

In this section we want to use the result of the previous section in order to obtain information on the Grassmann envelope of $M_{2,1}(F)$.

We denote by G the Grassmann algebra of countable rank over F. Recall that G is generated by the set $\{e_1, e_2, \ldots\}$ satisfying the relations

$$e_i e_j = -e_j e_i,\ i, j = 1, 2, \ldots$$

Let $G^{(0)}$ be the subspace of G generated by the monomials in the e_i's of even length and $G^{(1)}$ the subspace generated by the monomials in the e_i's of odd length. Then $G = G^{(0)} \oplus G^{(1)}$ is a $\mathbb{Z}_2$-graded algebra. For a $\mathbb{Z}_2$-graded algebra $A = A^{(0)} \oplus A^{(1)}$, the algebra $G(A) = (A^{(0)} \otimes G^{(0)}) \oplus (A^{(1)} \otimes G^{(1)})$ is called the Grassmann envelope of A. The importance of this construction is due to the fact that any variety of associative algebras can be generated by the Grassmann envelope of a finite dimensional $\mathbb{Z}_2$-graded algebra [12, Theorem 1].

Also, by a theorem of Kemer, a very important role in the theory of varieties of algebras is played by the verbally prime algebras. Recall that an algebra A is verbally prime if its T-ideal of identities $Id(A)$ has the following property:

$$\text{if } I_1 I_2 \subseteq Id(A) \text{ then, either } I_1 \subseteq Id(A) \text{ or } I_2 \subseteq Id(A),$$

for any T-ideals I_1, I_2.

By the classification given in [12, Lemma 9], the Grassmann envelope of $M_{2,1}(F)$, denoted $M_{2,1}(G)$, is a verbally prime algebra. Hence

$$M_{2,1}(G) = \begin{pmatrix} G^{(0)} & G^{(0)} & G^{(1)} \\ G^{(0)} & G^{(0)} & G^{(1)} \\ G^{(1)} & G^{(1)} & G^{(0)} \end{pmatrix}.$$

Let f be a multilinear polynomial of degree n. Then f can be represented in the form

$$f = \sum_u \sum_{\sigma \in S_n} \alpha_{\sigma,u} u_1 z_{\sigma(1)} u_2 z_{\sigma(2)} \cdots z_{\sigma(n)} u_{n+1},$$

where the u_i's are words in only even variables, $u = (u_1, \ldots, u_{n+1})$ and $\alpha_{\sigma,u} \in F$. Put

$$f^* = \sum_u \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \alpha_{\sigma,u} u_1 z_{\sigma(1)} \cdots z_{\sigma(n)} u_{n+1}.$$

If I is a T_2-ideal, we denote by I^* the T_2-ideal generated by the sets $(I \cap V_n)^*_{n \geqslant 1}$. There exist results relating the algebra $M_{2,1}(F)$ to its Grassmann envelope $M_{2,1}(G)$. In particular, the T_2-ideal of graded identities of $M_{2,1}(F)$ is related to the T_2-ideal of graded identities of $M_{2,1}(G)$ by the following [12, Lemma 4].

Lemma 11. *Let I be the T_2-ideal of graded identities of $M_{2,1}(F)$. Then I^* is the T_2-ideal of graded identities of $M_{2,1}(G)$.*

Another result relating their graded cocharacters is the following [2, Lemma 7].

Lemma 12. *If the nth graded cocharacter of $M_{2,1}(F)$ has the following decomposition*

$$\chi_n^{gr}(M_{2,1}(F)) = \sum m_{\lambda,\mu}\chi_{\lambda,\mu}$$

then

$$\chi_n^{gr}(M_{2,1}(G)) = \sum m_{\lambda,\mu}\chi_{\lambda,\mu'},$$

where μ' is the conjugate partition of μ.

Then by an easy application of Theorem 10, we have.

Theorem 13. *The nth graded cocharacter of $M_{2,1}(G)$ is*

$$\chi_n^{gr}(M_{2,1}(G)) = \sum_{\substack{|\lambda|+|\mu|=n \\ h(\lambda)\leqslant 5,\, h(\mu)\leqslant 4}} m_{\lambda,\mu}\chi_{\lambda,\mu'},$$

where $m_{\lambda,\mu} = 0$ if and only if (λ, μ) is equal to $((1^4), \emptyset)$ or to $((1^5), (1))$ or to $(\lambda, \emptyset)$ with $h(\lambda) = 5$.

5. Classifying the graded identities of degree $\leqslant 5$

In this section we determine all the graded polynomial identities of $M_{2,1}(F)$ of degree $\leqslant 5$. Observe that the graded identities of $M_{2,1}(F)$ in even variables are the same as the ordinary identities of $M_2(F)$, since $Id(M_{2,1}(F)^{(0)}) = Id(M_2(F) \oplus F) = Id(M_2(F)) \cap Id(F) = Id(M_2(F)$. Moreover, since char $F = 0$, we may restrict ourselves to consider only multilinear polynomials.

By making suitable evaluations it is easy to check that, given any multilinear polynomial $g(y_1, \dots, y_k, z_1, \dots, z_t)$, with $k + t \leqslant 4$, then g is a graded identity of $M_{2,1}(F)$ if and only if $g = \alpha St_4(y_1, \dots, y_4)$ for some $\alpha \in F$. We record this in the following.

Remark 14. If f is a graded identity of $M_{2,1}(F)$ of degree $\leqslant 4$, then

$$f = \alpha St_4(y_1, \dots, y_4)$$

for some $\alpha \in F$.

In order to describe the identities of degree 5 we shall use the representation theory of $GL_5 \times GL_5$.

Let $\lambda \vdash r$ and $\mu \vdash 5 - r$. We denote by $T(\lambda, \mu)$ the set of all pairs (T_λ, T_μ) of standard Young tableaux and by $d_{\lambda,\mu}$ its cardinality. If d_λ (resp. d_μ) denotes the number of standard λ-tableaux given by the hook formula [16] (resp. μ-tableaux) then

$$d_{\lambda,\mu} = \binom{n}{r} d_\lambda d_\mu.$$

We shall use the following result [7, Proposition 1].

Proposition 15. *Let $\lambda \vdash r$ and $\mu \vdash n-r$. The highest weight vector $f_{\lambda,\mu}$ can be expressed uniquely as a linear combination of the polynomials f_{T_λ,T_μ} with T_λ and T_μ standard tableaux.*

Let $Id^{gr}(M_{2,1}(F)) \cap F_5^5 \cong \sum \alpha_{\lambda,\mu} W^{\lambda,\mu}$, be the decomposition of the $GL_5 \times GL_5$-module $Id^{gr}(M_{2,1}(F)) \cap F_5^5$ into irreducibles where $W^{\lambda,\mu}$ is the irreducible submodule corresponding to the pair of partitions (λ, μ) generated by the highest weight vector $f_{\lambda,\mu}$.

Since T_2-ideals are completely homogeneous, we will study the modules $Id^{gr}(M_{2,1}(F)) \cap W_{i,j}$, where $W_{i,j}$ is the subspace of F_5^5 consisting of all homogeneous polynomials of total degree i in the even variables $y_1, \dots, y_5$ and of total degree j in the odd variables $z_1, \dots, z_5$, $i + j = 5$.

In the next lemmas we deduce the decompositions of the $GL_5 \times GL_5$-modules

$$Id^{gr}(M_{2,1}(F)) \cap W_{i,j}, \quad i, j = 1, \dots, 5, \quad i + j = 5,$$

into a direct sum of irreducible submodules. For each such decomposition we also exhibit a complete list of highest weight vectors.

Given an irreducible $GL_5 \times GL_5$-module $W^{\lambda,\mu}$ corresponding to the pair of partitions (λ, μ), we shall denote by $f'_{\lambda,\mu}, f''_{\lambda,\mu}, \dots$ distinct highest weight vectors corresponding to the same pair of partitions (λ, μ).

We also adopt the convention that the symbols $\bar{}$ and $\tilde{}$ indicate alternation on a given set of variables. Thus, for instance the notation $\overline{y_1}\tilde{z_1}y_4\overline{y_2}\tilde{z_2}\overline{y_3}$ indicates the polynomial

$$\sum_{\substack{\sigma \in S_3 \\ \tau \in S_2}} (\operatorname{sign} \sigma)(\operatorname{sign} \tau) y_{\sigma(1)} z_{\tau(1)} y_4 y_{\sigma(2)} z_{\tau(2)} y_{\sigma(3)}.$$

In particular $\overline{x_i}\,\overline{x_j} = [x_i, x_j]$.

In order to determine the decomposition of a fixed space $Id^{gr}(M_{2,1}(F)) \cap W_{i,j}$ into irreducibles, we proceed as follows.

We need to determine all the highest weight vectors $f_{\lambda,\mu}$ which are identities for $M_{2,1}(F)$. To this end, we observe that, by the previous proposition, any such $f_{\lambda,\mu}$ can be written uniquely as a linear combination $\sum_{i=1}^{d_{\lambda,\mu}} \alpha_i \omega_i$ of highest weight vectors ω_i corresponding to pairs of standard tableaux. Hence we start by writing down explicitly all the polynomials ω_i.

We view the coefficients α_i as unknowns and we evaluate $\sum_{i=1}^{d_{\lambda,\mu}} \alpha_i \omega_i$ into 3×3 graded generic matrices by imposing that it must be a graded polynomial identity of $M_{2,1}(F)$. In this way we obtain a homogeneous linear system of nine equations (corresponding to the nine entries of the resulting matrix) in $d_{\lambda,\mu}$ unknowns α_i. This system can be completely solved by making enough evaluations of the graded generic matrices in $M_{2,1}(F)$. We then prove that the graded polynomials so obtained are graded identities of $M_{2,1}(F)$. This way we obtain all the highest weight vectors $f_{\lambda,\mu}$ in $W_{i,j}$ which are identities for $M_{2,1}(F)$. In case we obtain several different highest weight vectors corresponding to the same pair of partitions (λ, μ), we check that they are linearly independent. We obtain in this way the multiplicity of the corresponding $GL_5 \times GL_5$-module.

By applying the above scheme we obtain the proof of the following lemmas.

Lemma 16. *The following decomposition holds*

$$Id^{gr}(M_{2,1}(F)) \cap W_{0,5} = W^{\emptyset,(4,1)} \oplus 2W^{\emptyset,(3,1^2)} \oplus W^{\emptyset,(2^2,1)} \oplus 2W^{\emptyset,(2,1^3)} \oplus W^{\emptyset,(1^5)}.$$

Moreover a complete list of corresponding highest weight vectors is given by the following:

$$\begin{aligned}
&f_{\emptyset,(4,1)} = [z_1^2, z_1z_2z_1],\\
&f'_{\emptyset,(3,1^2)} = \bar{z}_1z_1\bar{z}_2z_1\bar{z}_3, \quad f''_{\emptyset,(3,1^2)} = z_3\bar{z}_1z_3\bar{z}_2z_3,\\
&f_{\emptyset,(2^2,1)} = \bar{z}_1\tilde{z}_1\bar{z}_2\tilde{z}_2\bar{z}_3,\\
&f'_{\emptyset,(2,1^3)} = \bar{z}_1z_1\bar{z}_2\bar{z}_3\bar{z}_4, \quad f''_{\emptyset,(2,1^3)} = \bar{z}_1\bar{z}_2\bar{z}_3z_1\bar{z}_4,\\
&f_{\emptyset,(1^5)} = St_5(z_1, z_2, z_3, z_4, z_5).
\end{aligned}$$

Lemma 17

$$Id^{gr}(M_{2,1}(F)) \cap W_{1,4} = W^{(1),(4)} \oplus 3W^{(1),(2^2)} \oplus 2W^{(1),(2,1^2)} \oplus 2W^{(1),(1^4)}.$$

A complete list of corresponding highest weight vectors is given by the following:

$$\begin{aligned}
&f_{(1),(4)} = [zyz, z^2],\\
&f'_{(1),(2^2)} = [y, \bar{z}_1\tilde{z}_1\bar{z}_2\tilde{z}_2], \qquad f''_{(1),(2^2)} = \bar{z}_1y\bar{z}_2[z_1, z_2] - [z_1, z_2]\bar{z}_1y\bar{z}_2,\\
&f'''_{(1),(2^2)} = \bar{z}_1y\tilde{z}_1\bar{z}_2\tilde{z}_2 - \bar{z}_1\tilde{z}_1\bar{z}_2y\tilde{z}_2,\\
&f'_{(1),(2,1^2)} = [y, \bar{z}_1z_1\bar{z}_2\bar{z}_3 + St_3(z_1, z_2, z_3)z_1],\\
&f''_{(1),(2,1^2)} = \bar{z}_1yz_1\bar{z}_2\bar{z}_3 + \bar{z}_1y\bar{z}_2\bar{z}_3z_1 - \bar{z}_1z_1\bar{z}_2y\bar{z}_3 - \bar{z}_1\bar{z}_2\bar{z}_3yz_1,\\
&f'_{(1),(1^4)} = [y, St_4(z_1, z_2, z_3, z_4)], \quad f''_{(1),(1^4)} = \bar{z}_1y\bar{z}_2\bar{z}_3\bar{z}_4 - \bar{z}_1\bar{z}_2\bar{z}_3y\bar{z}_4.
\end{aligned}$$

Lemma 18

$$Id^{gr}(M_{2,1}(F)) \cap W_{2,3} = W^{(2),(1^3)} \oplus 3W^{(1^2),(1^3)} \oplus 2W^{(2),(2,1)} \oplus 4W^{(1^2),(2,1)} \oplus W^{(2),(3)} \oplus W^{(1^2),(3)}.$$

A complete list of corresponding highest weight vectors is given by the following:

$$
\begin{aligned}
&f_{(2),(1^3)} = \bar{z}_1\bar{z}_2 y\bar{z}_3 y - \bar{z}_1 y\bar{z}_2 y\bar{z}_3 - ySt_3(z_1, z_2, z_3)y + y\bar{z}_1 y\bar{z}_2\bar{z}_3,\\
&f'_{(1^2),(1^3)} = St_3(z_1, z_2, z_3)[y_1, y_2] + \bar{z}_1[y_1, y_2]\bar{z}_2\bar{z}_3,\\
&f''_{(1^2),(1^3)} = \bar{z}_1\bar{z}_2\tilde{y}_1\bar{z}_3\tilde{y}_2 + \bar{z}_1\tilde{y}_1\bar{z}_2\tilde{y}_2\bar{z}_3 - \tilde{y}_1\bar{z}_1\bar{z}_2\bar{z}_3\tilde{y}_2 + \tilde{y}_1\bar{z}_1\tilde{y}_2\bar{z}_2\bar{z}_3,\\
&f'''_{(1^2),(1^3)} = \bar{z}_1\bar{z}_2[y_1, y_2]\bar{z}_3 + [y_1, y_2]\bar{z}_1\bar{z}_2\bar{z}_3,\\
&f'_{(2),(2,1)} = y\bar{z}_1 yz_1\bar{z}_2 - y\bar{z}_1 z_1\bar{z}_2 y - \bar{z}_1 yz_1 y\bar{z}_2 + \bar{z}_1 z_1 y\bar{z}_2 y,\\
&f''_{(2),(2,1)} = y[z_1, z_2]z_1 y - [z_1, z_2]yz_1 y - y\bar{z}_1 y\bar{z}_2 z_1 + \bar{z}_1 y\bar{z}_2 yz_1,\\
&f'_{(1^2),(2,1)} = [y_1, y_2]\bar{z}_1 z_1\bar{z}_2 + \bar{z}_1 z_1[y_1, y_2]\bar{z}_2,\\
&f''_{(1^2),(2,1)} = \tilde{y}_1\bar{z}_1\tilde{y}_2 z_1\bar{z}_2 - \tilde{y}_1\bar{z}_1 z_1\bar{z}_2\tilde{y}_2 + \bar{z}_1\tilde{y}_1 z_1\tilde{y}_2\bar{z}_2 + \bar{z}_1 z_1\tilde{y}_1\bar{z}_2\tilde{y}_2,\\
&f'''_{(1^2),(2,1)} = \tilde{y}_1\bar{z}_1\tilde{y}_2\bar{z}_2 z_1 + \bar{z}_1\tilde{y}_1\bar{z}_2\tilde{y}_2 z_1 - \tilde{y}_1[z_1, z_2]z_1\tilde{y}_2 + [z_1, z_2]\tilde{y}_1 z_1\tilde{y}_2,\\
&f^{IV}_{(1^2),(2,1)} = \bar{z}_1[y_1, y_2]z_1\bar{z}_2 + \bar{z}_1 z_1\bar{z}_2[y_1, y_2],\\
&f_{(2),(3)} = yzyz^2 - yz^3 y + [z, zyzy],\\
&f_{(1^2),(3)} = \tilde{y}_1 z\tilde{y}_2 z^2 - \tilde{y}_1 z^3\tilde{y}_2 + z\tilde{y}_1 z\tilde{y}_2 z + z^2\tilde{y}_1 z\tilde{y}_2.
\end{aligned}
$$

Lemma 19

$$Id^{gr}(M_{2,1}(F)) \cap W_{3,2} = W^{(2,1),(2)} \oplus W^{(2,1),(1^2)} \oplus 2W^{(1^3),(2)} \oplus 2W^{(1^3),(1^2)}.$$

A complete list of corresponding highest weight vectors is given by the following:

$$
\begin{aligned}
&f_{(2,1),(2)} = [z[y_1, y_2]z, y_1]\\
&f_{(2,1),(1^2)} = [\bar{z}_1[y_1, y_2]\bar{z}_2, y_1],\\
&f'_{(1^3),(2)} = \tilde{y}_1 z\tilde{y}_2\tilde{y}_3 z - z\tilde{y}_1\tilde{y}_2 z\tilde{y}_3,\\
&f''_{(1^3),(2)} = [St_3(y_1, y_2, y_3), z^2] - \tilde{y}_1\tilde{y}_2 z^2\tilde{y}_3 + \tilde{y}_1 z^2\tilde{y}_2\tilde{y}_3,\\
&f'_{(1^3),(1^2)} = \tilde{y}_1\bar{z}_1\tilde{y}_2\tilde{y}_3\bar{z}_2 - \bar{z}_1\tilde{y}_1\tilde{y}_2\bar{z}_2\tilde{y}_3,\\
&f''_{(1^3),(1^2)} = [St_3(y_1, y_2, y_3), [z_1, z_2]] - \tilde{y}_1\tilde{y}_2[z_1, z_2]\tilde{y}_3 + \tilde{y}_1[z_1, z_2]\tilde{y}_2\tilde{y}_3.
\end{aligned}
$$

Lemma 20

$$Id^{gr}(M_{2,1}(F)) \cap W_{4,1} = 3W^{(1^4),(1)} \oplus W^{(2,1^2),(1)} \oplus W^{(2^2),(1)}.$$

A complete list of corresponding highest weight vectors is given by the following:

$$
\begin{aligned}
&f'_{(1^4),(1)} = zSt_4(y_1, y_2, y_3, y_4), \quad f''_{(1^4),(1)} = \tilde{y}_1\tilde{y}_2 z\tilde{y}_3\tilde{y}_4,\\
&f'''_{(1^4),(1)} = St_4(y_1, y_2, y_3, y_4)z,\\
&f_{(2,1^2),(1)} = \tilde{y}_1\tilde{y}_2 zy_1\tilde{y}_3 - \tilde{y}_1\tilde{y}_2 z\tilde{y}_3 y_1,\\
&f_{(2^2),(1)} = [y_1, y_2]z[y_1, y_2].
\end{aligned}
$$

Lemma 21

$$Id^{gr}(M_{2,1}(F)) \cap W_{5,0} = W^{(3,2),\emptyset} \oplus 3W^{(2,1^3),\emptyset} \oplus W^{(2^2,1),\emptyset} \oplus W^{(1^5),\emptyset}.$$

A complete list of corresponding highest weight vectors is given by the following:

$$\begin{aligned}
&f_{(3,2),\emptyset} = [y_1, [y_1, y_2]^2],\\
&f'_{(2,1^3),\emptyset} = \tilde{y}_1 y_1 \tilde{y}_2 \tilde{y}_3 \tilde{y}_4 + \tilde{y}_1 \tilde{y}_2 \tilde{y}_3 y_1 \tilde{y}_4,\\
&f''_{(2,1^3),\emptyset} = \tilde{y}_1 \tilde{y}_2 y_1 \tilde{y}_3 \tilde{y}_4, \quad f'''_{(2,1^3),\emptyset} = St_4(y_1, y_2, y_3, y_4) y_1,\\
&f_{(2^2,1),\emptyset} = \overline{y}_1 \tilde{y}_1 \overline{y}_2 \tilde{y}_2 \overline{y}_3 - \overline{y}_1 \tilde{y}_1 \overline{y}_2 \overline{y}_3 \tilde{y}_2 + 2\overline{y}_1 \overline{y}_2 \tilde{y}_1 \overline{y}_3 \tilde{y}_2 - 3\overline{y}_1 \overline{y}_2 [y_1, y_2] \overline{y}_3\\
&\qquad -St_3(y_1, y_2, y_3)[y_1, y_2],\\
&f_{(1^5),\emptyset} = St_5(y_1, y_2, y_3, y_4, y_5).
\end{aligned}$$

The decompositions given in the previous lemmas can be applied in order to find a minimal list of graded identities of $M_{2,1}(F)$ up to degree 5. In fact we can now prove the following theorem.

Theorem 22. *Let* $f = f(y_1, \ldots, y_k, z_1, \ldots, z_t) \in Id^{gr}(M_{2,1}(F))$ *be a graded identity of degree* $\leqslant 5$. *Then* f *is a consequence of the following polynomials*:

1) $\overline{z}_1 z_4 \overline{z}_2 z_5 \overline{z}_3$,
2) $z_3 \overline{z}_1 z_3 \overline{z}_2 z_3$,
3) $[y, \overline{z}_1 \tilde{z}_2 \overline{z}_3 \tilde{z}_4]$,
4) $[z_1 y z_2, z_3 z_4] + [z_3 y z_4, z_1 z_2]$,
5) $\overline{z}_1 z_2 \overline{z}_3 [y_1, y_2] + \overline{z}_1 [y_1, y_2] z_2 \overline{z}_3$,
6) $\overline{z}_1 z_2 [y_1, y_2] \overline{z}_3 + [y_1, y_2] \overline{z}_1 z_2 \overline{z}_3$,
7) $y_1 z_1 y_2 z_2 z_3 + z_1 z_2 y_1 z_3 y_2 - y_1 z_1 z_2 z_3 y_2 - z_1 y_2 z_2 y_1 z_3$,
8) $[z_1 [y_1, y_2] z_2, y_3]$,
9) $[y_1, y_2] z [y_3, y_4]$,
10) $St_4(y_1, y_2, y_3, y_4)$,
11) $[[y_1, y_2]^2, y_1]$.

Proof. If f is a graded identity of degree $\leqslant 4$ then, by Remark 14, it follows that f is a consequence of $St_4(y_1, y_2, y_3, y_4)$ and we are done.

Suppose now that f is a graded identity of degree 5. Since F is infinite, all the multihomogeneous components of f are still graded identity for $M_{2,1}(F)$. Hence we may assume that f is multihomogeneous. Thus $f \in W_{i,j}$, for some $i, j \leqslant 5$.

If $f \in W_{5,0}$ then f is a graded identity of $M_{2,1}(F)$ in only even variables and by Remark 3 and [3], f is a consequence of 10) and 11).

Suppose now that $f \in Id^{gr}(M_{2,1}(F)) \cap W_{i,j}$, $i \neq 5$. Since every element of $Id^{gr}(M_{2,1}(F)) \cap W_{i,j}$ is a consequence of the highest weight vectors of the previous lemmas, it is enough to check that any highest weight vector follows from these polynomials.

By changing name to the variables it is clear that the highest weight vector $f_{\emptyset,(4,1)}$ follows from the polynomial in 2), which coincides with $f''_{\emptyset,(3,1^2)}$. Similarly $f'_{\emptyset,(3,1^2)}$, $f_{\emptyset,(2^2,1)}$, $f'_{\emptyset,(2,1^3)}$, $f''_{\emptyset,(2,1^3)}$ and $f_{\emptyset,(1^5)}$ follow from 1).

It is clear, by renaming the variables, that $f_{(1),(4)}$ and $f'_{(1),(2^2)}$ follow from the identities in 4) and 3) respectively. Now, the polynomial $f''_{(1),(2,1^2)}$ can be written as

$$f''_{(1),(2,1^2)} = \bar{z}_1 y\tilde{z}_1\bar{z}_2\tilde{z}_3 - \bar{z}_1\tilde{z}_1\bar{z}_2 y\tilde{z}_3 + \bar{z}_1\tilde{z}_1\bar{z}_3 y\tilde{z}_2 - \bar{z}_1 y\tilde{z}_1\bar{z}_3\tilde{z}_2.$$

Notice that, the following equality holds

$$\begin{aligned}\bar{z}_1 y\tilde{z}_2\bar{z}_3\tilde{z}_4 - \bar{z}_1\tilde{z}_2\bar{z}_3 y\tilde{z}_4 = {} & [z_1 y z_2, z_3 z_4] + [z_3 y z_4, z_1 z_2] - [z_3 y z_2, z_1 z_4] \\ & -[z_1 y z_4, z_3 z_2],\end{aligned}$$

and this implies that $f''_{(1),(2,1^2)}$ is a consequence of the identity in 4). From the above equality it also follows that the polynomials

$$f'''_{(1),(2^2)} = \bar{z}_1 y\tilde{z}_1\bar{z}_2\tilde{z}_2 - \bar{z}_1\tilde{z}_1\bar{z}_2 y\tilde{z}_2 \quad \text{and} \quad f''_{(1),(1^4)} = \bar{z}_1 y\bar{z}_2\bar{z}_3\bar{z}_4 - \bar{z}_1\bar{z}_2\bar{z}_3 y\bar{z}_4$$

are consequences of the identity in 4). By linearizing it follows that the polynomial $f''_{(1),(2^2)} = [\bar{z}_1 y\bar{z}_2, \tilde{z}_1\tilde{z}_2]$ is a consequences of 4). If we write

$$f'_{(1),(2,1^2)} = [y, \bar{z}_1\tilde{z}_1\bar{z}_2\tilde{z}_3 + \bar{z}_1\tilde{z}_2\bar{z}_3\tilde{z}_1],$$

then it is also clear that the polynomials $f'_{(1),(1^4)}$ and $f'_{(1),(2,1^2)}$ are consequences of 3).

Consider now the space $Id^{gr}(M_{2,1}(F)) \cap W_{2,3}$. Clearly the highest weight vectors

$$f_{(2),(1^3)},\ f''_{(1^2),(1^3)},\ f'_{(2),(2,1)},\ f''_{(2),(2,1)},\ f'''_{(1^2),(2,1)},\ f_{(2),(3)},\ f_{(1^2),(3)} \text{ and } f''_{(1^2),(2,1)}$$

follow from 7). Also it is easily seen that the highest weight vectors $f'_{(1^2),(1^3)}$ and $f^{IV}_{(1^2),(2,1)}$ follow from 5) and $f'''_{(1^2),(1^3)}$ and $f'_{(1^2),(2,1)}$ follow from 6).

If we now consider the space $Id^{gr}(M_{2,1}(F)) \cap W_{3,2}$, then, at once it follows that the highest weight vectors

$$f_{(2,1),(2)},\ f_{(2,1),(1^2)},\ f'_{(1^3),(2)} = -[z\tilde{y}_1\tilde{y}_2 z, \tilde{y}_3] \quad \text{and} \quad f'_{(1^3),(1^2)} = -[\bar{z}_1\tilde{y}_1\tilde{y}_2\bar{z}_2, \tilde{y}_3]$$

are consequences of 8), while $f''_{(1^3),(2)}$ and $f''_{(1^3),(1^2)}$ are consequences of 10).

Finally consider the highest weight vectors of Lemma 20. Clearly $f'_{(1^4),(1)}$ and $f'''_{(1^4),(1)}$ follow from 10) while $f_{(2^2),(1)}$ and $f''_{(1^4),(1)}$ follow from 9). Consider now $f_{(2,1^2),(1)}$. Since we can write

$$f_{(2,1^2),(1)} = \tilde{y}_1\tilde{y}_2 z y_1\tilde{y}_3 - \tilde{y}_1\tilde{y}_2 z\tilde{y}_3 y_1 = [y_1, y_2]z[y_1, y_3] - [y_1, y_3]z[y_1, y_2]$$

then it is clear that it follows from 9). The proof is therefore complete. □

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